

4/MTH-253 Syllabus-2023

2026

(May-June)

FYUP : 4th Semester Examination

MATHEMATICS

(Matrix Theory and Vector Spaces)

(MTH-253)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions.

Answer **four** questions, taking **one** from each Unit

UNIT—I

1. (a) (i) Show that if $AB = A$ and $BA = B$, then A and B are idempotent matrices.

(ii) Show that

$$\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B) \quad 2+2=4$$

- (b) Prove that every Hermitian matrix A can be uniquely expressed as $B+iC$, where B is real and symmetric and C is real and skew-symmetric matrix, and $i = \sqrt{-1}$.

4

(2)

- (c) Find the inverse of the matrix

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & -1 \\ 1 & 1 & 4 \end{bmatrix}$$

using elementary operations.

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- (d) Reduce the matrix

$$\begin{bmatrix} 0 & 1 & -3 \\ 2 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}$$

to a row-reduced echelon matrix and hence obtain its rank.

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2. (a) Given

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

prove that $A^3 - 3A^2 - 7A - 11I_3 = 0$.

Hence find A^{-1} .

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- (b) Show that the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

is nilpotent.

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(3)

- (c) Prove that the diagonal elements of a skew-symmetric matrix are all 0.

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- (d) If A, B, C are 2×2 matrices such that $AB = AC$, then does it follow that $B = C$? Justify your answer.

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- (e) Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 1 \end{bmatrix}$$

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UNIT—II

3. (a) Determine if the following system of equations is consistent and if so, then find the solution : $3+3=6$

$$x - y + 2z = 4$$

$$3x + y + 4z = 6$$

$$x + y + z = 1$$

- (b) Compute the determinantal rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 \\ 2 & -1 & -3 & 5 \\ -1 & 8 & -6 & -1 \end{bmatrix}$$

using the definition of determinantal rank.

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- (c) Show that if A is an $n \times n$ matrix, then

$$\text{Adj}(\text{Adj } A) = |A|^{n-2} \cdot A \quad 4$$

- (d) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 4 & -3 \\ 1 & -1 & 2 \end{bmatrix}$$

using adjoint of the matrix. 5

4. (a) Find the value of k for which the system

$$x + 2y + 3z = 0$$

$$2x + 3y + kz = 0$$

$$3x + 4y + 8z = 0$$

has infinitely many solutions and trivial solution. 5

- (b) Solve the system of equations

$$x - y + z = 0$$

$$2x + 3y - 5z = 7$$

$$3x - 4y - 2z = -1$$

by Crammer's rule. 5

- (c) Find the adjoint of the matrix

$$\begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & 1 \\ 1 & -1 & -2 \end{bmatrix} \quad 4$$

- (d) Find the solutions of homogeneous system of equations

$$x - 2y + z = 0$$

$$x + y = z$$

$$3x + 6y = 5z \quad 5$$

UNIT—III

5. (a) Prove that a subset W of a vector space V over a field F is a subspace of V if and only if $\forall a, b \in F$ and $u, v \in W$, $au + bv \in W$. Hence show that the set

$$W = \{(x, y, z) \in \mathbb{R}^3 : 2x - 3y + z = 0\}$$

is a subspace of $\mathbb{R}^3$ over $\mathbb{R}$. 5+3=8

- (b) If F is the field of real numbers, show that the vectors $(1, 1, 0, 0)$, $(0, 1, -1, 0)$ and $(0, 0, 0, 3)$ in $F^{(4)}$ are linearly independent over F . 4

- (c) Show that the vectors $(2, 1, 4)$, $(1, -1, -2)$ and $(3, 1, -2)$ form a basis for $\mathbb{R}^3$. 4

- (d) Prove that if V is a finite dimensional vector space of dimension n over the field F , then any set of n linearly independent vectors in V forms a basis of V . 3

6. (a) If a_1, a_2 and a_3 are fixed elements of a field F , then prove that the set W of all ordered triads (x_1, x_2, x_3) of elements of F , such that $a_1x_1 + a_2x_2 + a_3x_3 = 0$ is a subspace of $V_3(F)$. 4
- (b) If W_1 and W_2 are subspaces of the vector space $V(F)$, then prove that
 (i) $W_1 + W_2$ is a subspace of $V(F)$ and
 (ii) $L(W_1 \cup W_2) = W_1 + W_2$. 3+4=7
- (c) Prove that if two vectors are linearly dependent, then one of them is a scalar multiple of the other. 3
- (d) Prove that two finite dimensional vector spaces $V(F)$ and $U(F)$ over a field F are isomorphic if and only if $\dim U = \dim V$. 5

UNIT—IV

7. (a) Show that a mapping $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by
- $$T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$$
- is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$. Find the range (T), rank (T), null space and nullity (T). 3+1+1+2+1=8

- (b) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation defined as $T(x, y) = (x + y, x - y)$. Find the matrix of T relative to the basis $\{(1, 1), (1, -1)\}$. 3
- (c) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation defined as $T(x, y) = (y, -x)$. Is T invertible? If so, find the inverse of T . 2+3=5
- (d) Let $\{v_1, v_2, \dots, v_n\}$ be a basis of V and let T be linear transformation on V such that $T(v_1) = v_2, T(v_2) = v_3, \dots, T(v_n) = v_1$. Show that $T^n = I$, the identity transformation. 3
8. (a) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 6 \\ 0 & 0 & -3 \end{bmatrix} \quad 3+4=7$$

- (b) Verify Cayley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and hence find A^{-1} . 3+3=6

- (c) Let V be a vector space over the field F . Let T be a linear transformation from V into V . Prove that $\text{range}(T) \cap \text{null}(T) = 0$ if and only if $T(T(v)) = 0 \Rightarrow T(v) = 0$. 3
- (d) Let U and V be vector spaces over the field F . Let T be a linear transformation from U into V . Prove that $\ker(T)$ is a subspace of U . 3

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